Extrapolating Treatment Effects in Multi-Cutoff Regression Discontinuity Designs
Supplemental Appendix

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Gonzalo Vazquez-Bare§

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Abstract

This supplemental appendix includes a description of RD empirical papers with multiple cutoffs, further extensions and generalizations of our main methodological results, and omitted proofs and derivations. In addition, we outline an extension of our methods to the RD local randomization framework introduced by Cattaneo et al. (2015) and Cattaneo et al. (2017). Finally, R and Stata replication files are available at https://sites.google.com/site/rdpackages/replication/.

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SA-1  **Empirical Examples of Multi-Cutoff RD Designs**

Table SA-1 collects over 30 empirical papers in social, behavioral and biomedical sciences, where RD designs with multiple cutoffs are present. In most of these papers, the multi-cutoff feature of the RD designs was not exploited, but rather a pooling-and-normalizing approach was taken to conduct the empirical analysis (see Cattaneo et al., 2016, for methodological background).

### Table SA-1: Empirical Papers Employing RD Designs with Multiple Cutoffs.

<table>
<thead>
<tr>
<th>Citation</th>
<th>Place</th>
<th>Running Variable</th>
<th>Outcome Variable</th>
<th>Cutoffs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abdulkadroglu et al. (2017)</td>
<td>US</td>
<td>Test scores</td>
<td>Test scores</td>
<td>Many</td>
</tr>
<tr>
<td>Angrist and Lavy (1999)</td>
<td>Israel</td>
<td>Cohort size</td>
<td>Test scores</td>
<td>2</td>
</tr>
<tr>
<td>Behaghel, Crépon and Sédillot (2008)</td>
<td>France</td>
<td>Age</td>
<td>Layoff rates</td>
<td>Many</td>
</tr>
<tr>
<td>Berk and de Leeuw (1999)</td>
<td>U.S.</td>
<td>Prison score</td>
<td>Re-conviction</td>
<td>4</td>
</tr>
<tr>
<td>Black, Gallow and Smith (2007)</td>
<td>US</td>
<td>Training eligibility score</td>
<td>Job training and aid</td>
<td>Many</td>
</tr>
<tr>
<td>Brodlolo and Nannicini (2012)</td>
<td>Brazil</td>
<td>Population</td>
<td>Federal transfers</td>
<td>Many</td>
</tr>
<tr>
<td>Buddelmeyer and Skoufas (2004)</td>
<td>Mexico</td>
<td>Poverty score</td>
<td>Education attainment</td>
<td>Many</td>
</tr>
<tr>
<td>Canton and Blom (2004)</td>
<td>Mexico</td>
<td>Eligibility score</td>
<td>College outcomes</td>
<td>Many</td>
</tr>
<tr>
<td>Card and Giuliano (2016)</td>
<td>US</td>
<td>Eligibility score</td>
<td>Test score</td>
<td>Many</td>
</tr>
<tr>
<td>Chay, McEwan and Urquiola (2005)</td>
<td>Chile</td>
<td>Eligibility score</td>
<td>School aid</td>
<td>13</td>
</tr>
<tr>
<td>Chen and Shapiro (2004)</td>
<td>US</td>
<td>Prison score</td>
<td>Rearrest</td>
<td>5</td>
</tr>
<tr>
<td>Chen and Van der Klaauw (2008)</td>
<td>U.S.</td>
<td>Age</td>
<td>Disability awards</td>
<td>3</td>
</tr>
<tr>
<td>Clark (2009)</td>
<td>UK</td>
<td>Majority vote</td>
<td>Test scores</td>
<td>Many</td>
</tr>
<tr>
<td>Crost, Felter and Klaauw (2014)</td>
<td>Phillipines</td>
<td>Poverty index</td>
<td>Conflict</td>
<td>22</td>
</tr>
<tr>
<td>Dell and Querubin (2017)</td>
<td>Vietnam</td>
<td>Geographic regions</td>
<td>Military strategy by location</td>
<td>Many</td>
</tr>
<tr>
<td>Dobkin and Ferreira (2010)</td>
<td>U.S.</td>
<td>Birthday</td>
<td>Education attainment</td>
<td>3</td>
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<tr>
<td>Garibaldi et al. (2012)</td>
<td>Italy</td>
<td>Income</td>
<td>Graduation</td>
<td>12</td>
</tr>
<tr>
<td>Goodman (2008)</td>
<td>U.S.</td>
<td>Test score</td>
<td>Scholarship</td>
<td>Many</td>
</tr>
<tr>
<td>Hjalmarsson (2009)</td>
<td>U.S.</td>
<td>Adjudication score</td>
<td>Re-conviction</td>
<td>2</td>
</tr>
<tr>
<td>Hockstra (2009)</td>
<td>US</td>
<td>Test and grades</td>
<td>Earnings</td>
<td>Many</td>
</tr>
<tr>
<td>Hoxby (2000)</td>
<td>U.S.</td>
<td>Cohort size</td>
<td>Test scores</td>
<td>Many</td>
</tr>
<tr>
<td>Kne (2003)</td>
<td>U.S.</td>
<td>GPA</td>
<td>College attendance</td>
<td>Many</td>
</tr>
<tr>
<td>Klaśnja and Titimuk (2017)</td>
<td>Brazil</td>
<td>Margin of victory</td>
<td>Incumbency advantage</td>
<td>Many</td>
</tr>
<tr>
<td>Kirkeboen, Leuven and Mogstad (2016)</td>
<td>Norway</td>
<td>Test scores</td>
<td>Earnings</td>
<td>Many</td>
</tr>
<tr>
<td>Litschig and Morrison (2013)</td>
<td>Brazil</td>
<td>Population</td>
<td>Poverty reduction</td>
<td>17</td>
</tr>
<tr>
<td>Spenukh and Toniatti (2018)</td>
<td>US</td>
<td>County borders</td>
<td>Advertising and vote shares</td>
<td>Many</td>
</tr>
<tr>
<td>Snider and Williams (2015)</td>
<td>US</td>
<td>Distance from airports</td>
<td>Airfares</td>
<td>Many</td>
</tr>
<tr>
<td>Urquiola (2006)</td>
<td>Bolivia</td>
<td>Cohort size</td>
<td>Test scores</td>
<td>2</td>
</tr>
<tr>
<td>Urquiola and Verhoogen (2009)</td>
<td>Chile</td>
<td>Cohort size</td>
<td>Test scores</td>
<td>3</td>
</tr>
<tr>
<td>Van der Klaauw (2002)</td>
<td>U.S.</td>
<td>Aid score</td>
<td>Financial aid</td>
<td>Many</td>
</tr>
<tr>
<td>Van der Klaauw (2008)</td>
<td>U.S.</td>
<td>Poverty Score</td>
<td>School aid</td>
<td>Many</td>
</tr>
</tbody>
</table>

**Note:** “Many” refers to examples where either a large number of cutoff points are present or a continuum of cutoff points can be generated (e.g., the cutoff is a continuous random variable). This table excludes a large number of political science and related applications reported in Cattaneo et al. (2016, Supplemental Appenedix).
Table SA-2: Results for ACCES loan eligibility on Post-Education Enrollment in restricted sample

<table>
<thead>
<tr>
<th></th>
<th>Estimate</th>
<th>Bw</th>
<th>Eff. N</th>
<th>Robust BC Inference</th>
</tr>
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<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>p-value</td>
</tr>
<tr>
<td><strong>RD effects</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C = -850$</td>
<td>0.061</td>
<td>85.0</td>
<td>85</td>
<td>0.282</td>
</tr>
<tr>
<td>$C = -571$</td>
<td>0.169</td>
<td>136.3</td>
<td>133</td>
<td>0.103</td>
</tr>
<tr>
<td>Weighted</td>
<td>0.121</td>
<td>218</td>
<td></td>
<td>0.056</td>
</tr>
<tr>
<td>Pooled</td>
<td>0.073</td>
<td>161.5</td>
<td>307</td>
<td>0.254</td>
</tr>
<tr>
<td><strong>Naive difference</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu_\ell(-650)$</td>
<td>0.728</td>
<td>239.4</td>
<td>440</td>
<td></td>
</tr>
<tr>
<td>$\mu_\ell(-650)$</td>
<td>0.706</td>
<td>131.2</td>
<td>202</td>
<td></td>
</tr>
<tr>
<td>Difference</td>
<td>0.021</td>
<td></td>
<td></td>
<td>0.634</td>
</tr>
<tr>
<td><strong>Bias</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu_\ell(-850)$</td>
<td>0.560</td>
<td>58.0</td>
<td>57</td>
<td></td>
</tr>
<tr>
<td>$\mu_\ell(-850)$</td>
<td>0.667</td>
<td>144.2</td>
<td>230</td>
<td></td>
</tr>
<tr>
<td>Difference</td>
<td>-0.106</td>
<td></td>
<td></td>
<td>0.017</td>
</tr>
<tr>
<td><strong>Extrapolation</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tau_\ell(-650)$</td>
<td>0.128</td>
<td></td>
<td></td>
<td>0.022</td>
</tr>
</tbody>
</table>

Notes. Local polynomial regression estimation with MSE-optimal bandwidth selectors and robust bias corrected inference. See Calonico et al. (2014) and Calonico et al. (2018) for methodological details, and Calonico et al. (2017) and Cattaneo et al. (2020) for implementation. “Eff. N” indicates the effective sample size, that is, the sample size within the MSE-optimal bandwidth. “Bw” indicates the MSE-optimal bandwidth.

**SA-2 Additional Empirical Results**

In this section we explore the sensitivity of our empirical results when restricting the sample to the period 2007-2010. This check allows us to reduce the heterogeneity of the sample due to variation over time. The results are shown in Table SA-2. We find overall similar results, with a positive and significant extrapolated effect of 12.8 percentage points.

**SA-3 Simulation Setup**

We provide further details on the simulation setup used in the paper. Potential outcome regression functions are generated in the following way:

$$
\mu_{0,h}(x) = r_p(x)' \gamma, \quad \mu_{0,\ell}(x) = \mu_{0,h}(x) + \Delta, \quad \mu_{1,c}(x) = \mu_{0,c}(x) + \tau
$$

3
Table SA-3: Segments and Corresponding Parameters in Figure ??

<table>
<thead>
<tr>
<th>Segment</th>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ab$</td>
<td>$\tau_l(\bar{x}) = \mu_{1,l}(\bar{x}) - \mu_{0,l}(\bar{x})$</td>
<td>Extrapolation parameter of interest</td>
</tr>
<tr>
<td>$bc$</td>
<td>$B(\bar{x}) = \mu_{0,l}(\bar{x}) - \mu_{0,h}(\bar{x})$</td>
<td>Control facing $l$ vs. control facing $h$, at $X_i = \bar{x}$</td>
</tr>
<tr>
<td>$ac$</td>
<td>$\tau_l(\bar{x}) + B(\bar{x}) = \mu_{1,l}(\bar{x}) - \mu_{0,h}(\bar{x})$</td>
<td>Treated facing $l$ vs. control facing $h$, at $X_i = \bar{x}$</td>
</tr>
<tr>
<td>$cd$</td>
<td>$B(\ell) = \mu_{0,l}(\ell) - \mu_{0,h}(\ell)$</td>
<td>Control facing $l$ vs. control facing $h$, at $X_i = \ell$</td>
</tr>
</tbody>
</table>

where $r_p(x)$ is a $p$-th order polynomial basis for $x$. We set $p = 4$. The value of $\gamma$, given below, is chosen by running a regression of the observed outcome for the high-cutoff on a fourth order polynomial using data from our empirical application. The observed variables are generated according to the following model:

$$
X_i \sim \text{Uniform}(-1000, -1)
$$

$$
\{C_i\}_{i=1}^N \sim \text{FixedMargins}(N, N_\ell)
$$

$$
D_i = \mathbb{1}(X_i \geq C_i)
$$

$$
Y_i = \mu_{0,h}(X_i) + \tau D_i + \Delta \mathbb{1}(C_i = \ell) + \text{Normal}(0, \sigma^2)
$$

where $\text{FixedMargins}(N, N_\ell)$ denotes a fixed-margins distribution that assigns $N_\ell$ observations out of the total $N$ sample to face cutoff $\ell$ and the remaining $N - N_\ell$ ones to face cutoff $h$. 


Proceeding as above, we set the parameters of the data generating process as follows:

\[ \gamma = (-14.089, -0.074, -1.372e(-4), -1.125e(-7), -3.444e(-11))' \]
\[ \Delta = -0.14, \quad \tau = 0.19, \quad \sigma^2 = 0.3^2 \]
\[ N \in \{1000, 2000, 5000\}, \quad N_\ell = N/2 \]
\[ \ell = -850, \quad \tilde{h} = -571, \quad \tilde{x} = -650. \]

All these parameters were estimated using local polynomial and related methods.

**SA-4 Extensions and Generalizations**

We briefly discuss some extensions and generalizations of our main extrapolation results.

**SA-4.1 Conditional-on-covariates Constant Bias**

Following Abadie (2005), we can relax the constant bias assumption to hold conditionally on observable characteristics. Let the bias term conditional on a vector of observable covariates \( Z_i = z \in \mathbf{Z} \) be denoted by \( B(x, z) = \mathbb{E}[Y_i(0)|X_i = x, Z_i = z, C_i = \ell] - \mathbb{E}[Y_i(0)|X_i = x, Z_i = z, C_i = \bar{h}] \). Let \( p(x, z) = \mathbb{P}(C_i = \ell|X_i = x, Z_i = z) \) denote the low-cutoff propensity score and \( p(x) = \mathbb{P}(C_i = \ell|X_i = x) \).

We impose the following conditions:

**Assumption SA-1 (Ignorable Constant Bias)**

1. **Conditional bias**: \( B(\ell, z) = B(x, z) \) for all \( x \in (\ell, \bar{h}) \) and for all \( z \in \mathbf{Z} \).
2. **Common support**: \( \delta < p(x, z) < 1 - \delta \) for some \( \delta > 0, x \in \mathbf{X} \) and for all \( z \in \mathbf{Z} \).
3. **Continuity**: \( p(x, z) \) is continuous in \( x \) for all \( z \in \mathbf{Z} \).

Part 1 states that the selection bias is equal across cutoffs after conditioning on a covariates, part 2 is the usual assumption ruling out empty cells defined by the covariates \( Z_i \), and part
3 assumes that the propensity score is continuous in the running variable. Then, letting $S_i = \mathbb{1}(C_i = \ell)$, we have the following result, which is proven in the supplemental appendix.

**Theorem SA-1 (Covariate-Adjusted Extrapolation)** Under Assumption 1 and SA-1,

\[
\tau_\ell(\bar{x}) = \mathbb{E} \left[ \frac{Y_i S_i}{p(\bar{x})} \mid X_i = h \right] - \mathbb{E} \left[ \frac{Y_i(1 - S_i)}{1 - p(\bar{x}, Z_i)} \cdot \frac{p(\bar{x}, Z_i)}{p(\bar{x})} \mid X_i = x \right] + \mathbb{E} \left[ \frac{Y_i(1 - S_i) p(\bar{x}, Z_i)}{p(\bar{x})(1 - p(\ell, Z_i))} \cdot \frac{f_{X|Z}(\ell|Z_i)}{f_{X|Z}(\ell|Z_i)} \cdot \frac{f_X(\ell)}{f_X(h)} \mid X_i = \ell \right] - \lim_{x \to \ell^-} \mathbb{E} \left[ \frac{Y_i S_i p(\bar{x}, Z_i)}{p(\bar{x}) p(\ell, Z_i)} \cdot \frac{f_{X|Z}(\ell|Z_i)}{f_{X|Z}(\ell|Z_i)} \cdot \frac{f_X(\ell)}{f_X(h)} \mid X_i = x \right].
\]

This result is somewhat notationally convoluted, but it is straightforward to implement. It gives a precise formula for extrapolating RD treatment effects for values of the score above the cutoff $\ell$, based on a conditional-on-$Z_i$ constant bias assumption. All the unknown quantities in Theorem SA-1 can be replaced by consistent estimators thereof, under appropriate regularity conditions.

**SA-4.2 Non-Constant Bias**

Although the constant bias restriction (Assumption 2) is intuitive and allows for a helpful analogy with the difference-in-differences design, the RD setup leads for a natural extension via local polynomial extrapolation. The score is a continuous random variable and, under additional smoothness conditions of the bias function $B(x)$, we can replace the constant bias assumption with the assumption that $B(x)$ can be approximated by a polynomial expansion of $B(\ell)$ around $x \in (\ell, h)$.

For example, using a polynomial of order one, we can approximate $B(x)$ at $\bar{x}$ as

\[
B(\bar{x}) \approx B(\ell) + \hat{B}(\ell) \cdot [\bar{x} - \ell]
\]

where $\hat{B}(x) = \hat{\mu}_{0,\ell}(x) - \hat{\mu}_{0,h}(x)$ and $\hat{\mu}_{d,c}(x) = \partial \mu_{d,c}(x) / \partial x$. This shows that the constant bias assumption $B(x) = B(\ell)$ can be seen as a special case of the above approximation, where
the first derivatives of $\mu_{0,l}(x)$ and $\mu_{0,h}(x)$ are assumed equal to each other at $x = \bar{x}$. In contrast, the approximation in (SA-1) allows these derivatives to be different, and corrects the extrapolation at $\bar{x}$ using the difference between them at the point $\ell$.

This idea allows, for instance, for different slopes of the control regression functions at $\bar{x}$, leading to $B(\bar{x}) \neq B(\ell)$. The linear adjustment in expression (SA-1) provides a way to correct the bias term to account for the difference in slopes at the low cutoff $\ell$. This represents a generalization of the constant assumption, which allows the intercepts of $\mu_{0,l}(x)$ and $\mu_{0,h}(x)$ to differ, but does not allow their difference to be a function of $x$. It is straightforward to extend this reasoning to employ higher order polynomials to approximate $B(\bar{x})$, at the cost of a stronger smoothness and extrapolation assumptions.

**Assumption SA-2 (Polynomial-in-Score Bias)**

1. **Smoothness:** $\mu_{d,c}(x) = \mathbb{E}[Y_i(d) \mid X_i = x, C_i = c]$ are $p$-times continuously differentiable at $x = c$ for all $c \in C$, $d = 0, 1$ and for some $p \in \{0, 1, 2, \ldots\}$.

2. **Polynomial Approximation:** there exists a $p \in \{0, 1, 2, \ldots\}$ such that, for $x \in (\ell, h)$

$$B(x) = \sum_{s=0}^{p} \frac{1}{s!} B^{(s)}(\ell) \cdot [x - \ell]^s$$

where $B^{(s)}(x) = \mu^{(s)}_{0,l}(x) - \mu^{(s)}_{0,h}(x)$ and $\mu^{(s)}_{d,c}(x) = \partial^s \mu_{d,c}(x)/\partial x^s$.

The main extrapolation result in can be generalized as follows.

**Theorem SA-2** Under Assumption SA-2, for $\bar{x} \in (\ell, h)$,

$$\tau_\ell(\bar{x}) = \mu_\ell(\bar{x}) - \left[ \mu_h(\bar{x}) + \sum_{s=0}^{p} \frac{1}{s!} B^{(s)}(\ell) \cdot [x - \ell]^s \right].$$

This result establishes valid extrapolation of the RD treatment effect away from the low cutoff $\ell$. This time the extrapolation is done via adjusting for not only the constant difference between the two control regression functions but also their higher-order derivatives.
Heuristically, this result justifies approximating the control regression functions by a higher-order polynomial, local to the cutoff, and then using the additional information about higher-derivatives to extrapolate the treatment effects.

SA-4.3 Many Cutoffs

All the identification results in the main paper hold for any number of cutoffs $J \geq 2$. The key issue to be assessed in this case is whether the constant bias assumption holds for all subpopulations. When this is the case, $\tau_c(x)$ is overidentified (for $x \geq c$), since there are many control groups that can be used to identify this parameter. In this setting, joint estimation can be performed using fixed effects models, as explained in the next section.

Furthermore, when more than two cutoffs are available, more control regression functions are therefore available for extrapolation. These could be combine to extrapolate or, alternatively, each of them could be used to extrapolate the RD treatment effect and only after these treatment effects could be combined. We relegate this interesting problem for future work.

SA-4.4 Multi-Scores and Geographic RD Designs

In many applications, RD designs include multiple scores (e.g., Papay et al., 2011; Reardon and Robinson, 2012; Keele and Titiunik, 2015). Examples include treatments assigned based on not exceeding a given threshold on two different scores (Figure SA-1(a)) or based on being one side of some generic boundary (Figure SA-1(b)). While these designs induce a continuum of cutoff points, it is usually better to analyze them using a finite number of cutoffs along the boundary determining treatment assignment.

For example, in Figure SA-1 we illustrate two settings with three chosen cutoff points (A, B, and C). In panel (a), the three cutoffs correspond to “extremes” over the boundary, while in panel (b) the cutoff points are chosen towards the “center”. Once these cutoffs points are chosen, the analysis can proceed as discussed in the main paper. Usually, the
multidimensionality of the problem is reduced by relying on some metric that maps the multi-score feature of the design to a unidimensional problem. For instance, $X_i$ is usually taken to be a measure of distance relative to the desired cutoff point. Once this mapping is constructed, all the ideas and methods discussed in the paper can be extended and applied to extrapolate RD treatment effects across cutoff points in multi-score RD designs.

**SA-5 Relationship with Fixed Effects Models**

Consider a separable model for the potential outcomes and only two cutoffs $\ell < h$:

$$Y_i(0) = g_0(X_i) + \gamma_0 \mathbb{1}(C_i = h) + \epsilon_{0i}, \quad \mathbb{E}[\epsilon_{0i} | X_i, C_i] = 0$$

$$Y_i(1) = g_1(X_i) + \gamma_1 \mathbb{1}(C_i = h) + \epsilon_{1i}, \quad \mathbb{E}[\epsilon_{1i} | X_i, C_i] = 0$$

This model assumes that $X_i, C_i$ and the error terms are separable. A key implication of separability is the absence of interaction between the cutoff and the score. In other words,
changing the cutoff only shifts the conditional expectation function without affecting the slope. The above model implies:

$$
\tau_c(x) = \mathbb{E}[Y_i(1) - Y_i(0) \mid X_i = x, C_i = c] = g_1(x) - g_0(x) + (\gamma_1 - \gamma_0)1(c = h)
$$

Also, let $$D_i = \mathbb{1}(X_i \geq C_i)$$, $$S_i = \mathbb{1}(C_i = h)$$, $$\gamma \equiv \gamma_0$$ and $$\delta \equiv \gamma_1 - \gamma_0$$, so that defining the observed outcome in the usual way, $$Y_i = D_iY_i(1) + (1 - D_i)Y_i(0)$$, we get:

$$
Y_i = g_0(X_i) + \gamma S_i + (g_1(X_i) - g_0(X_i))D_i + \delta D_i \times S_i + \varepsilon_i
$$

where $$\varepsilon_i = \varepsilon_0D_i + \varepsilon_1(1 - D_i)$$ and $$\mathbb{E}[\varepsilon_i \mid X_i, C_i] = 0$$.

Although restrictive, this separable model nests several particular cases that are commonly used in RD estimation. For instance, if we assume:

$$
g_0(x) = \alpha_0 + \beta_0 x, \quad g_1(x) = \alpha_1 + \beta_1 x,
$$

the model reduces to:

$$
Y_i = \alpha_0 + \beta_0 X_i + (\alpha_1 - \alpha_0)D_i + (\beta_1 - \beta_0)X_i \times D_i + \gamma S_i + \delta D_i \times S_i + \varepsilon_i
$$

which is the usual linear model with an interaction between the score and the treatment, with two additional terms that account for the presence of two cutoffs.

The assumption of separability is sufficient for the selection bias to be constant across cutoffs. More precisely,

$$
\mathbb{E}[Y_i(0) \mid X_i = x, C_i = \ell] = g_0(x) \\
\mathbb{E}[Y_i(0) \mid X_i = x, C_i = h] = g_0(x) + \gamma_0 \\
\Rightarrow B(x) = \gamma_0, \quad \forall x
$$
In this setting, consider the case of many cutoffs is relatively straightforward. With $C_i \in \{c_0, c_1, ..., c_J\} := C$, the potential outcomes can be defined as:

$$Y_i(0) = g_0(X_i) + \sum_{j=1}^{J} \gamma_{0j} \mathbb{1}(C_i = c_j) + \varepsilon_i$$

$$Y_i(1) = g_1(X_i) + \sum_{j=1}^{J} \gamma_{1j} \mathbb{1}(C_i = c_j) + \varepsilon_i$$

with observed outcome:

$$Y_i = g_0(X_i) + \sum_{j=1}^{J} \gamma_{0j} S_{ij} + (g_1(X_i) - g_0(X_i)) D_i + \sum_{j=1}^{J} \delta_j D_i \times S_{ij} + \varepsilon_i$$

where $S_{ij} = \mathbb{1}(C_i = c_j)$ and $\delta_j = \gamma_{1j} - \gamma_{0j}$. As before, the above model implies that the selection bias does not depend on the running variable:

$$B(x, c_j, c_k) = \gamma_{0j} - \gamma_{0k}$$

The equation for the observed outcome can be rewritten as:

$$Y_{ij} = \gamma_j + g(X_{ij}) + \theta_j D_{ij} + \tau_j (X_{ij}) D_{ij} + \varepsilon_{ij}$$

where $Y_{ij}$ is the outcome for unit $i$ exposed to cutoff $c_j$ and $D_{ij}$ is equal to one if unit $i$ exposed to cutoff $c_j$ is treated, $D_{ij} = \mathbb{1}(X_i \geq c_j) \times \mathbb{1}(C_i = c_j)$. This model is similar to a one-way fixed effects model.
SA-5.1 Example: linear case

Using the above “fixed-effects notation”, suppose \( g_0 \) and \( g_1 \) are linear:

\[
Y_{ij}(0) = \gamma_{0j} + \beta_0 X_{ij} + \varepsilon_{ij}^0 \\
Y_{ij}(1) = \gamma_{1j} + \beta_1 X_{ij} + \varepsilon_{ij}^1
\]

where \( Y_{ij}(d) \) is the potential outcome of unit \( i \) facing cutoff \( c_j \) under treatment \( d \). As before, the fact that the slopes change with the treatment status but not with the cutoff is implied by separability between the score and the cutoff indicator. The observed outcome is:

\[
Y_{ij} = \gamma_{0j} + \beta_0 X_{ij} + (\gamma_{1j} - \gamma_{0j}) D_{ij} + (\beta_{1j} - \beta_0) X_{ij} \times D_{ij} + \varepsilon_{ij}
\]

with \( \varepsilon_{ij} = \varepsilon_{ij}^1 D_{ij} + \varepsilon_{ij}^0 (1 - D_{ij}) \).

Reparameterizing the model gives the estimating equation:

\[
Y_{ij} = \gamma_j + \beta X_{ij} + \delta_j D_{ij} + \theta_j X_{ij} \times D_{ij} + \varepsilon_{ij}
\]

which is a linear model including cutoff fixed effects, the running variable, the treatment variable with cutoff-varying coefficients and the interaction between the score and the treatment. In other words, this is the standard linear RD specification, but with different intercepts and slopes at each cutoff. Note that the coefficients for \( X_{ij} \) do not vary with \( j \). This captures the restriction that the slopes of the conditional expectations under no treatment are the same across cutoffs and also leads to a straightforward specification test.

Under the linear specification, the treatment effect evaluated at \( c_k \) for units facing cutoff \( c_j \) with \( c_k > c_j \) is given by:

\[
\tau_{c_j}(\bar{x}) = \gamma_{1j} - \gamma_{0j} + (\beta_{1j} - \beta_0) \bar{x}
\]

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and can be estimated as:

\[ \hat{\tau}_{c_j}(\bar{x}) = \hat{\delta}_j + \hat{\theta}_j \bar{x} \]

or simply as \( \hat{\delta}_j \) if the running variable is appropriately re-centered.

Clearly, assuming a linear specification would in principle allow one to estimate the effects not only at the cutoffs but also at any other point in the range of the score. The treatment effects away from the cutoffs, however, are not identified nonparametrically and their identification relies purely on the functional form assumption, which can make them less credible.

The specification test mentioned above for the linear case consists on running the regression:

\[ Y_{ij} = \gamma_j + \beta_j X_{ij} + \delta_j D_{ij} + \theta_j X_{ij} \times D_{ij} + \varepsilon_{ij} \]

and testing:

\[ H_0 : \beta_1 = \beta_2 = \ldots = \beta_J. \]

**SA-6 Local Randomization Methods**

Our core extrapolation ideas can be adapted to the local randomization RD framework of Cattaneo et al. (2015) and Cattaneo et al. (2017). In this alternative framework, only units whose scores lay within a fixed (and small) neighborhood around the cutoff are considered, and their potential outcomes are regarded as fixed. The key source of randomness comes from the treatment assignment mechanism—the probability law placing units in control and treatment groups, and consequently the analysis proceeds as if the RD design was a randomized experiment within the neighborhood. See Rosenbaum (2010) and Imbens and Rubin (2015) for background on classical Fisher and Neyman approaches to the analysis of experiments.

As in the continuity-based approach, the Multi-Cutoff RD design can be analyzed using
local randomization methods by either normalizing-and-pooling all cutoffs or by studying each cutoff separately. However, to extrapolate away from an RD cutoff (i.e., outside the small neighborhood where local randomization is assumed to hold), further strong identifying assumptions are needed. To discuss these additional assumptions, we first formalize the local randomization (LR) framework for extrapolation in a Multi-Cutoff RD design.

Recall that \( C = \{l, h\} \) for simplicity. Let \( N_x \) be a (non-empty) LR neighborhood around \( x \in [l, h] \), that is, 
\[
N_x = [x - w, x + w], \quad w > 0,
\]
where the (half) window length may be different for each center point \( x \). The other neighborhoods discussed below are defined analogously; for example, \( N_{\bar{x}} = [\bar{x} - w, \bar{x} + w] \) for some \( w > 0 \), possibly different across neighborhoods. Let also \( y_{ic}(d, x) \) be a non-random potential outcome for unit \( i \) when facing cutoff \( c \) with treatment assignment \( d \) and running variable \( x \). Consequently, in this Multi-Cutoff RD LR framework each unit has non-random potential outcomes \( \{y_{il}(0, x), y_{il}(1, x), y_{ih}(0, x), y_{ih}(1, x)\} \), for each \( x \in \mathcal{X} \). The observed outcome is 
\[
Y_i = y_{iC_i}(0, X_i)\mathbb{I}(X_i < C_i) + y_{iC_i}(1, X_i)\mathbb{I}(X_i \geq C_i) \quad \text{where} \quad C_i \in C.
\]
Recall that our goal is to extrapolate the RD treatment effect to a point \( \bar{x} \in (l, h] \) on the support of the running variable.

**Assumption SA-3 (LR Extrapolation)**

(i) For all \( i \) such that \( X_i \in \mathcal{N}_l \),
\[
\{y_{il}(0, x), y_{il}(1, x), y_{ih}(0, x)\} = \{y_{il}(0, \ell), y_{il}(1, \ell), y_{ih}(0, \ell)\}
\]
for all \( x \in \mathcal{N}_l \), and are non-random. Furthermore, the treatment assignment mechanism is known.
(ii) For $\bar{x} \in (l, h]$ such that $N_l \cap N_{\bar{x}} = \emptyset$ and for all $i$ such that $X_i \in N_{\bar{x}},$

$$\{y_{id}(0, x), y_{id}(1, x), y_{idh}(0, x)\} = \{y_{id}(0, \bar{x}), y_{id}(1, \bar{x}), y_{idh}(0, \bar{x})\}$$

for all $x \in N_{\bar{x}}$, and are non-random. Furthermore, the treatment assignment mechanism is known.

(iii) There exists a constant $\Delta \in \mathbb{R}$ such that:

$$y_{id}(0, \ell) = y_{idh}(0, \ell) + \Delta, \quad \text{for all } i \text{ such that } X_i \in N_\ell,$$

and

$$y_{id}(0, \bar{x}) = y_{idh}(0, \bar{x}) + \Delta, \quad \text{for all } i \text{ such that } X_i \in N_{\bar{x}}.$$

Assumption SA-3(i) is analogous to Assumption 1 in Cattaneo et al. (2015) applied to the RD cutoff $c = \ell$, except for the presence of one additional potential outcome, $y_{idh}(0)$, which we will use for extrapolation in the Multi-Cutoff RD design. Likewise, Assumption SA-3(ii) postulates the existence of a LR neighborhood for the desired point of extrapolation $\bar{x}$. Finally, Assumption SA-3(iii) imposes a relationship between (the difference of) control potential outcomes in the LR neighborhood of $c = \ell$ and (the difference of) control potential outcomes in the LR neighborhood of $c = \bar{x}$, which we will use to impute the missing control potential outcomes for units exposed to the low cutoff $c = \ell$ but with scores within $N_{\bar{x}}$.

To conserve space and notation, we do not extend Assumption SA-3 to allow for regression adjustments within the LR neighborhoods as in Cattaneo et al. (2017), but we do include the corresponding results in our empirical application.

In the LR framework, extrapolation requires imputing both the assignment mechanism and the missing control potential outcomes $y_{id}(0, \bar{x})$ within $N_{\bar{x}}$. As a consequence, extrapolation beyond the standard LR neighborhood $N_\ell$ requires very strong assumptions. Assumption SA-3 provides a set of conditions that lead to valid extrapolation. The parameter of interest
is the average effect of the treatment for units with \( X_i \in \mathcal{N}_x \):

\[
\tau_{LR} = \frac{1}{N_x} \sum_{X_i \in \mathcal{N}_x} (y_{i\ell}(1, \bar{x}) - y_{i\ell}(0, \bar{x}))
\]

where \( N_x \) is the number of units inside the window \( \mathcal{N}_x \) around \( \bar{x} \). Under Assumption SA-3, this parameter equals: 

\[
\frac{1}{N_x} \sum_{X_i \in \mathcal{N}_x} (y_{i\ell}(1, \bar{x}) - y_{ih}(0, \bar{x})) - \Delta,
\]

which is identifiable from the data. We implement this result as follows. First, we construct an estimate of \( \Delta \) as the difference-in-means for control units facing cutoffs \( \ell \) and \( h \) with \( X_i \in \mathcal{N}_\ell \), which we denote by \( \hat{\Delta} \):

\[
\hat{\Delta} = \bar{Y}_{\ell}(0, \ell) - \bar{Y}_h(0, \ell)
\]

where

\[
\bar{Y}_{\ell}(0, \ell) = \frac{1}{N_\ell(\ell)} \sum_{X_i \in \mathcal{N}_\ell} Y_i \mathbb{1}(C_i = \ell)(1 - D_i), \quad \bar{Y}_h(0, \ell) = \frac{1}{N_h(\ell)} \sum_{X_i \in \mathcal{N}_\ell} Y_i \mathbb{1}(C_i = h)
\]

and

\[
N_\ell(\ell) = \sum_{X_i \in \mathcal{N}_\ell} \mathbb{1}(C_i = \ell)(1 - D_i), \quad N_h(\ell) = \sum_{X_i \in \mathcal{N}_\ell} \mathbb{1}(C_i = h).
\]

Second, we estimate the treatment effects as:

\[
\hat{\tau}_{LR} = \bar{Y}_{\ell}(1, \bar{x}) - \bar{Y}_h(0, \bar{x}) - \hat{\Delta}
\]

where

\[
\bar{Y}_{\ell}(1, \bar{x}) = \frac{1}{N_\ell(\bar{x})} \sum_{X_i \in \mathcal{N}_x} Y_i \mathbb{1}(C_i = \ell), \quad \bar{Y}_h(0, \bar{x}) = \frac{1}{N_h(\bar{x})} \sum_{X_i \in \mathcal{N}_x} Y_i \mathbb{1}(C_i = h)
\]

and

\[
N_\ell(\bar{x}) = \sum_{X_i \in \mathcal{N}_x} \mathbb{1}(C_i = \ell), \quad N_h(\bar{x}) = \sum_{X_i \in \mathcal{N}_x} \mathbb{1}(C_i = h).
\]
Finally, for the assignment mechanism, we assume:

$$\mathbb{P}[C_i = h | X_i \in \mathcal{N}_c] = \frac{N_h(c)}{N_{\ell}(c) + N_h(c)}, \quad \forall \; i : X_i \in \mathcal{N}_c, \; c \in \{\ell, \bar{x}\}$$

and

$$\mathbb{P}[D_i = 0 | C_i = \ell, X_i \in \mathcal{N}_c] = \frac{N_{\ell}(0)}{N_{\ell}(\ell)}, \quad N_{\ell}(\ell) = \sum_{X_i \in \mathcal{N}_c} \mathbb{1}(C_i = \ell)$$

It is straightforward to show that under this assignment mechanism and Assumption SA-3, $\hat{\Delta}$ and $\hat{\tau}$ are unbiased for their corresponding parameters. Our approach is not the only way to develop LR methods for extrapolation, but for simplicity we focus on the above construction which mimics closely the continuity-based proposed in the previous sections.

In this setting, inference can be conducted using Fisherian randomization inference by permuting the cutoff indicator $\mathbb{1}(C_i = \ell)$ on the adjusted outcomes $Y^*_i = Y_i + \Delta \mathbb{1}(C_i = h)$ among units in $\mathcal{N}_h$. However, the inference procedure needs to account for the fact that $\Delta$ is unknown and needs to be estimated. We propose two alternatives to deal with this issue. The first one, suggested by Berger and Boos (1994), consists on constructing a $(1 - \eta)$-level confidence interval for $\Delta$, $S_\eta$, and defining the p-value:

$$p^*(\eta) = \sup_{\Delta \in S_\eta} p(\Delta) + \eta$$

which can be shown to be valid in the sense that $\mathbb{P}[p^*(\eta) \leq \alpha] \leq \alpha$.

Our second inference procedure, based on Neyman’s sampling approach, consists on using the standard normal distribution to approximate the distribution of the studentized statistic:

$$T = \frac{\hat{\tau}}{\sqrt{V_1 + V_\Delta}}$$

where $V_1$ is the estimated variance of the difference in means $\bar{Y}_{\ell}(1, \bar{x}) - \bar{Y}_h(0, \bar{x})$ and $V_\Delta$ is the estimated variance of $\hat{\Delta}$. 
SA-6.1 Empirical Application

We use our proposed LR methods to investigate the external validity of the ACCES program RD effects. As mentioned in the paper, our sample has observations exposed to two cutoffs, \( \ell = -850 \) and \( \bar{h} = -571 \). We begin by extrapolating the effect to the point \( \bar{x} = -650 \); our focus is thus the effect of eligibility for ACCES on whether the student enrolls in a higher education program for the subpopulation exposed to cutoff 850 when their SABER 11 score is 650.

Table SA-4 presents empirical results using the local randomization framework. We construct the neighborhoods \( N_\ell \) and \( N_{\bar{x}} \) using the 50 closest observations to the evaluation point of interest. To calculate \( p^*(\eta) \), we construct a 99 percent confidence interval for \( \Delta \) based on the normal approximation, which can be justified using large sample approximations in either a fixed potential outcomes model (Neyman) or a standard repeated sampling model (superpopulation). We estimate \( \hat{\tau}_{LR} \) using a constant model and using a linear adjustment (see Cattaneo et al., 2017, for details). Overall, the results are very similar to the ones obtained using the continuity-based approach. We find positive effects of around 20 percentage points that are significant at the 5 percent level using either Fisherian-based or Neyman-based inference.

To assess robustness of the LR methods, Figure SA-2 shows how the estimated effect and its corresponding randomization inference p-value change when varying the number of nearest neighbors used to construct \( N_\ell \) and \( N_{\bar{x}} \). The magnitude of the estimated effect remains stable when increasing the length of the window, particularly for the linear adjustment case which can help to reduce bias when the corresponding regression functions are not constant. In terms of inference, while the p-values we construct can be very conservative, we find significant effects at the 5 percent level when using around 45 observations in each neighborhood.
Table SA-4: Empirical Results under Local Randomization

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>(Y_t(0, \ell)) [-900, -850]</td>
<td>50</td>
<td>0.502</td>
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<td></td>
<td></td>
<td>0.527</td>
</tr>
<tr>
<td>(\bar{Y}_t(0, \ell)) [-881, -817]</td>
<td>50</td>
<td>0.706</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.707</td>
</tr>
<tr>
<td>(\Delta) 100</td>
<td>-0.204</td>
<td>0.000</td>
<td>0.000</td>
<td></td>
<td></td>
<td>0.000</td>
<td>0.021</td>
</tr>
<tr>
<td>(Y_t(1, \bar{x})) [-675, -626]</td>
<td>50</td>
<td>0.760</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.759</td>
</tr>
<tr>
<td>(\bar{Y}_t(0, \bar{x})) [-675, -625]</td>
<td>50</td>
<td>0.743</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.743</td>
</tr>
<tr>
<td>Diff 100</td>
<td>0.017</td>
<td>0.702</td>
<td>0.698</td>
<td></td>
<td></td>
<td>0.016</td>
<td>0.718</td>
</tr>
</tbody>
</table>

Notes. Estimated effect under the local randomization framework. Randomization inference p-values for \(\hat{\tau}_{LR}\) constructed using the Berger and Boos (1994) method. Neyman-based p-values constructed using a large-sample normal approximation. Estimates calculated using a constant model based on difference in means and a linear regression adjustment (see Cattaneo et al., 2017, for details). “Eff. N” indicates the effective sample size, that is, the sample size within the local randomization neighborhood.

Figure SA-2: Sensitivity Analysis for Local Randomization

Notes. The figure plots the estimated effect as a function of the number of nearest neighbors used around the cutoff for estimation. The left panel plots the estimates under a constant model. The right panel plots the estimates using a linear adjustment model. Hollow markers indicate p-value ≥ 0.15. Light gray markers indicate p-value < 0.15. Dark gray markers indicate p-value < 0.1. Black markers indicate p-value < 0.05. Randomization inference p-values constructed using the Berger and Boos (1994) method.
SA-7  Proofs and Derivations

SA-7.1  Proof of Theorem 1

Follows immediately under the stated assumptions by the derivation provided in the paper.

SA-7.2  Proof of Theorem 2

We omit the $i$ subscript to simplify notation. The average observed outcomes are:

$$
\mathbb{E}[Y|X = x, C = c] = \mathbb{E}[Y(1)D(x, c) + Y(0)(1 - D(x, c))|X = x, C = c] \\
= \mathbb{E}[(Y(1) - Y(0))D(x, c)|X = x, C = c] + \mathbb{E}[Y(0)|X = x, C = c]
$$

The difference in outcomes for units facing cutoffs $l < h$ at $X = x \in (l, h)$ is:

$$
\Delta(x) = \mathbb{E}[Y|X = x, C = l] - \mathbb{E}[Y|X = x, C = h] \\
= \mathbb{E}[(Y(1) - Y(0))D(x, l)|X = x, C = l] - \mathbb{E}[(Y(1) - Y(0))D(x, h)|X = x, C = h] \\
+ \mathbb{E}[Y(0)|X = x, C = l] - \mathbb{E}[Y(0)|X = x, C = h]
$$

Assuming parallel regression functions under no treatment, the double difference recovers:

$$
\Delta(x) - \Delta(l) \\
= \mathbb{E}[Y|X = x, C = l] - \mathbb{E}[Y|X = x, C = h] - \lim_{x \uparrow l} \mathbb{E}[Y|X = x, C = l] + \mathbb{E}[Y|X = l, C = h] \\
= \mathbb{E}[(Y(1) - Y(0))D(x, l)|X = x, C = l] - \mathbb{E}[(Y(1) - Y(0))D^{-}(l, l)|X = l, C = l] \\
- \{\mathbb{E}[(Y(1) - Y(0))D(x, h)|X = x, C = h] - \mathbb{E}[(Y(1) - Y(0))D(l, h)|X = l, C = h]\}
$$
where $D^-(l, l) = \lim_{x \uparrow l} D(x, l)$. Note that in a sharp design, $D(x, l) = 1$ and $D(x, h) = D(l, h) = D^-(l, l) = 0$ giving our previous result. After adding and subtracting,

$$\Delta(x) - \Delta(l)$$

$$= \mathbb{E}[Y|X = x, C = l] - \mathbb{E}[Y|X = x, C = h] - \lim_{x \uparrow l} \mathbb{E}[Y|X = x, C = l] + \mathbb{E}[Y|X = l, C = h]$$

$$= \mathbb{E}[(Y(1) - Y(0))(D(x, l) - D^-(l, l))|X = x, C = l]$$

$$+ \mathbb{E}[(Y(1) - Y(0))D^-(l, l)|X = x, C = l] - \mathbb{E}[(Y(1) - Y(0))D^-(l, l)|X = l, C = l]$$

$$- \{ \mathbb{E}[(Y(1) - Y(0))D(x, h)|X = x, C = h] - \mathbb{E}[(Y(1) - Y(0))D(l, h)|X = l, C = h] \}$$

Under one-sided non-compliance, units below the cutoff never get the treatment, so $D(x, h) = D(l, h) = D^-(l, l) = 0$. In this case,

$$\Delta(x) - \Delta(l) = \mathbb{E}[(Y(1) - Y(0))D(x, l)|X = x, C = l]$$

$$= \mathbb{E}[Y(1) - Y(0)|X = x, C = l, D(x, l) = 1] \mathbb{P}[D(x, l) = 1|X = x, C = l]$$

It follows that

$$\frac{\Delta(x) - \Delta(l)}{\mathbb{E}[D|X = x, C = l]} = \mathbb{E}[Y(1) - Y(0)|X = x, C = l, D(x, l) = 1]$$

which in this case equals the (local) average effect on the compliers.

**SA-7.3 Proof of Theorem SA-1**

Following analogous steps to the derivation for the unconditional case, we obtain that:

$$\mathbb{E}[\tau_i|X_i = \bar{x}, Z_i = z, C_i = \ell] = \mathbb{E}[Y_i|X_i = \bar{x}, Z_i = z, C_i = \ell]$$

$$- \mathbb{E}[Y_i|X_i = \bar{x}, Z_i = z, C_i = \ell] - B(\ell, z) \]
\[
\tau_\ell(\bar{x}) = E\{E[\tau_i | X_i = \bar{x}, Z_i, C_i = \ell] | X_i = \bar{x}, C_i = \ell}\}
\]

Define \(p(x, z) = P(C_i = \ell | X_i = x, Z_i = z)\) and \(S_i = 1(C_i = \ell)\). We have that:

\[
E[Y_i | X_i = \bar{x}, C_i = \ell, Z_i] = E\left[\frac{Y_i S_i}{p(\bar{x}, Z_i)} \bigg| X_i = \bar{x}, Z_i \right]
\]

\[
E[Y_i | X_i = \ell, C_i = h, Z_i] = E\left[\frac{Y_i (1 - S_i)}{1 - p(\ell, Z_i)} \bigg| X_i = \ell, Z_i \right]
\]

and similarly for the remaining two terms, we obtain, assuming that \(p(x, z)\) is continuous in \(x\) (and limits can be interchanged),

\[
E[\tau_i | X_i = \bar{x}, Z_i = z, C_i = \ell] = E\left[\frac{Y_i S_i}{p(\bar{x}, Z_i)} \bigg| X_i = \bar{x}, Z_i \right] - E\left[\frac{Y_i (1 - S_i)}{1 - p(\bar{x}, Z_i)} \bigg| X_i = \bar{x}, Z_i \right]
\]

\[
+ E\left[\frac{Y_i (1 - S_i)}{1 - p(\ell, Z_i)} \bigg| X_i = \ell, Z_i \right] - \lim_{x \to \ell^-} E\left[\frac{Y_i S_i}{p(\ell, Z_i)} \bigg| X_i = x, Z_i \right]
\]

To simplify the notation, let \(\tau_c(x, z) = E[\tau_i | X_i = x, C_i = c, Z_i = z]\), \(\tau_c(x) = E[\tau_i | X_i = x, C_i = c]\) and \(p(x) = P(C_i = \ell | X_i = x)\). By Bayes’ rule:

\[
\tau_\ell(\bar{x}) = \int \tau_\ell(\bar{x}, z) dF_{Z|X,C}(z | \bar{x}, \ell)
\]

\[
= \int \tau_\ell(\bar{x}, z) \frac{p(\bar{x}, z)}{p(\bar{x})} dF_{Z|X}(z | \bar{x})
\]

\[
= E\left[\tau_\ell(\bar{x}, Z_i) \frac{p(\bar{x}, Z_i)}{p(\bar{x})} \bigg| X_i = \bar{x}\right]
\]

Now replace \(\tau_\ell(\bar{x}, Z_i)\) with its observed counterpart and split the outer expectation into the four summands. We have that:

\[
E\left[E\left[\frac{Y_i S_i}{p(\bar{x}, Z_i)} \bigg| X_i = \bar{x}, Z_i \right] \frac{p(\bar{x}, Z_i)}{p(\bar{x})} \bigg| X_i = \bar{x}\right] = E\left[E\left[\frac{Y_i S_i}{p(\bar{x})} \bigg| X_i = \bar{x}, Z_i \right] \bigg| X_i = \bar{x}\right]
\]

\[
= E\left[\frac{Y_i S_i}{p(\bar{x})} \bigg| X_i = \bar{x}\right]
\]
Similarly, under regularity conditions to interchange limits and expectations,

\[
\begin{align*}
&\mathbb{E} \left[ \mathbb{E} \left[ \frac{Y_i(1 - S_i)}{1 - p(\ell, Z_i)} \bigg| X_i = \ell, Z_i \right] \mathbb{E} \left[ \frac{p(\bar{x}, Z_i)}{p(\bar{x})} \bigg| X_i = \bar{x} \right] \right] \\
&= \mathbb{E} \left[ \mathbb{E} \left[ \frac{Y_i(1 - S_i)}{1 - p(\ell, Z_i)} \cdot \frac{p(\bar{x}, Z_i)}{p(\bar{x})} \bigg| X_i = \ell, Z_i \right] \bigg| X_i = \bar{x} \right] \\
&= \mathbb{E} \left[ \mathbb{E} \left[ Y_i(1 - S_i) \cdot \frac{p(\bar{x}, Z_i)}{p(\bar{x})} \cdot \frac{f_{Z|X}(Z_i|\bar{x})}{f_{Z|X}(Z_i|\ell)} \bigg| X_i = \ell, Z_i \right] \bigg| X_i = \bar{x} \right] \\
&= \mathbb{E} \left[ \mathbb{E} \left[ Y_i(1 - S_i)p(\bar{x}, Z_i) \cdot \frac{f_{Z|X}(Z_i|\bar{x})}{f_{Z|X}(Z_i|\ell)} \frac{f_X(\ell)}{f_X(\bar{x})} \bigg| X_i = \ell, Z_i \right] \bigg| X_i = \bar{x} \right]
\end{align*}
\]

Putting all the results together,

\[
\tau_\ell(\bar{x}) = \mathbb{E} \left[ \frac{Y_iS_i}{p(\bar{x})} \bigg| X_i = \bar{x} \right] - \mathbb{E} \left[ \frac{Y_i(1 - S_i)}{1 - p(\bar{x}, Z_i)} \cdot \frac{p(\bar{x}, Z_i)}{p(\bar{x})} \bigg| X_i = x \right] \\
+ \mathbb{E} \left[ \frac{Y_i(1 - S_i)p(\bar{x}, Z_i)}{p(\bar{x})(1 - p(\ell, Z_i))} \cdot \frac{f_{Z|X}(Z_i|\bar{x})}{f_{Z|X}(Z_i|\ell)} \cdot \frac{f_X(\ell)}{f_X(\bar{x})} \bigg| X_i = \ell \right] \\
- \lim_{x \to \ell^-} \mathbb{E} \left[ \frac{Y_iS_i}{p(\bar{x})} \cdot \frac{f_{Z|X}(Z_i|\bar{x})}{f_{Z|X}(Z_i|\ell)} \cdot \frac{f_X(\ell)}{f_X(\bar{x})} \bigg| X_i = x \right].
\]
SA-7.4  Proof of Theorem **SA-2**

Follows immediately under the stated assumptions by the derivation provided in the paper.
References


——— (2020), “Analysis of Regression Discontinuity Designs with Multiple Cutoffs or Multiple Scores,” working paper.


